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TREATMENT OF BODY FORCES IN AXISYMMETRIC BOUNDARY ELEMENT DESIGN SENSITIVITY FORMULATION

SUNIL SAIGAL
Mechanical Engineering Department, Worcester Polytechnic Institute, Worcester, MA 01609, U.S.A.

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Abstract—This paper deals with the treatment of uniform and non-uniform body forces in the implicit-differentiation formulation for axisymmetric boundary element design sensitivity analysis. The particular integral concept is extended to obtain the sensitivities due to gravitational and centrifugal body forces of uniform type. For thermal body forces of non-uniform type, the sensitivities are obtained through the implicit-differentiation of a surface integral. The efficiency and the accuracy of the formulation are determined for a wide range of problems which include different axisymmetric geometries under centrifugal, gravitational and thermal body forces.

INTRODUCTION

The inclusion of body forces of the gravitational, centrifugal and thermal types has received a good deal of attention in boundary element research. The consideration of such forces is essential in the design of the high performance components such as, for example, fan and turbine disks in a gas turbine engine. Due to their critical performance requirements, optimal shapes are often desired for their configuration. The tools currently available for such optimal analysis do provide the information needed but at the expense of substantially increased computational resources. As more complex designs for even more stringent performance requirements evolve, it is essential to develop efficient tools for determining optimal configurations for these designs with due regard to the significant types of body forces.

The boundary element method (BEM) offers the possibility of being an efficient method for such iterative analysis as shape optimization. This is because of the reduced dimensionality of the problem since BEM requires only the surface of the body to be discretized as opposed to other available methods that require full domain discretization. While the earlier formulation of BEM by Cruse (1975) did require volume discretization for the treatment of body forces, Cruse *et al.* (1977) and Rizzo and Shippy (1977) used the field equations of the body force potential and the divergence theorem to reduce the volume integrals corresponding to the conservative body forces into an equivalent surface integral. Recently, Henry *et al.* (1987) and Pape and Banerjee (1987) developed a method based on particular integrals which requires neither volume nor surface integration for the treatment of uniform body forces.

A survey of the efforts in the area of sensitivity analysis was given by Mota Soares and Choi (1986). Some of the work concerning shape sensitivity analysis subsequent to this survey has included a finite difference formulation by Wu (1986), and implicit-differentiation formulations by Kane and Saigal (1988), Barone and Yang (1988), and Saigal *et al.* (1989a,b). Mukherjee and Chandra (1989) presented a boundary element design sensitivity formulation for materially nonlinear problems. None of these papers, however, have included the treatment of body forces in sensitivity analysis. The present paper deals with the development of a formulation for the computation of design sensitivities for axisymmetric continua under gravitational, centrifugal and thermal type body forces. The implicit-differentiation of the discretized boundary integral equations is performed leading to a set of system sensitivity equations. The sensitivity expressions for the particular integrals for gravitational and centrifugal body forces are presented. The thermoelastic sensitivity kernels are given for thermal type body forces. A set of test examples involving various axisymmetric designs are solved. These solutions are compared with exact analytical sensitivity solutions which are obtained by taking the material derivative of the corresponding analytical response expressions. The validity of the present formulations is established through a close agreement with exact analytical results. No previous results have been reported in the literature concerning design sensitivity analysis with body forces using the boundary element method.

DESIGN SENSITIVITY FORMULATION

Some basic equations of elasticity and boundary element analysis are first summarized. The equilibrium relationship for a homogeneous isotropic body subjected to a system of body forces is given by

$$\sigma_{ij,j} + \psi_i = 0 \tag{1}$$

where σ_{ij} is the stress tensor and ψ_i are the body forces. The stress and strain tensors including the effect of temperature variations are given by

$$\sigma_{ij} = \frac{E}{(1+v)} e_{ij} + \frac{vE}{(1+v)(1-2v)} \delta_{ij} e_{kk} - \frac{E}{(1-2v)} \delta_{ij} \alpha \phi \tag{2}$$

$$e_{ij} = \frac{(1+v)}{E} \sigma_{ij} - \frac{v}{E} \delta_{ij} \sigma_{kk} + \delta_{ij} \alpha \phi \tag{3}$$

where α is the coefficient of thermal expansion; ϕ is the temperature change from some reference state; v is Poisson's ratio; E is the modulus of elasticity; and e_{ij} is the strain tensor. Equation (1) via equations (2) and (3) can be written as

$$\mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + (\lambda + \mu) \frac{\partial^2 u_j}{\partial x_i \partial x_j} - \frac{E}{(1-2v)} \delta_{ij} \alpha \phi_{,j} + \psi_i = 0. \tag{4}$$

The term $\psi_i' = -\alpha E \phi_{,j}/(1-2v)$ can be considered as an equivalent body force due to temperature variation. Somigliana's identity for the displacements inside a body at point p due to tractions and displacements on the surface at the boundary point q is expressed in integral form as

$$u_i(p) = -\int_\Gamma T_{ij}(p,q) u_j(q)\, \mathrm{d}\Gamma + \int_\Gamma U_{ij}(p,q) t_j'(q)\, \mathrm{d}\Gamma + \int_\Omega U_{ij}(p,q) F_i(q)\, \mathrm{d}\Omega \tag{5}$$

where

$$F_i = \frac{-E\alpha}{(1-2v)} \phi_{,j} + \psi_i \tag{6}$$

$$t_j'(q) = t_j(q) + \frac{\alpha E}{(1-2v)} \phi n_j. \tag{7}$$

Using the divergence theorem, equation (5) can be expressed in the form given by Bakr (1986) as

$$u_i(p) = -\int_\Gamma T_{ij}(p,q)u_j(q)\,\mathrm{d}\Gamma + \int_\Gamma U_{ij}(p,q)t_j(q)\,\mathrm{d}\Gamma + \int_\Omega U_{ij}(p,q)\psi_i(q)\,\mathrm{d}\Omega - \frac{\alpha E}{(1-2\nu)}\int_\Omega U_{ij,j}(p,q)\phi\,\mathrm{d}\Omega. \quad (8)$$

U_{ij} and T_{ij} are the fundamental solutions for displacements and tractions, respectively; u_j and t_j are the boundary displacements and tractions, respectively; and Γ is the surface bounding the volume Ω of the body. The body forces ψ_i and the thermal effects due to temperature variation ϕ are taken into account through the volume integrals in equation (8). For certain types of body forces, the volume integrals shown in equation (8) need not be evaluated. The centrifugal and gravitational type body forces fall into this category. The design sensitivity formulation for such body forces is presented first. The volume integral for thermal effects can be reduced to a surface integral. The sensitivity formulation for such cases and the associated thermoelastic sensitivity kernels are presented next.

Gravitational and centrifugal type body force design sensitivity

The solution to equation (4) can be written as

$$u_i = u_i^c + u_i^p \quad (9)$$

where u_i^c is the complementary solution satisfying $L(u_i^c) = 0$, with $L = \mu\,\partial^2/\partial x_j\,\partial x_j + (\lambda+\mu)\times\partial^2/\partial x_i\,\partial x_j$, u_i^p is the particular integral solution satisfying $L(u_i^p)+\psi_i = 0$, ψ_i are the body forces of gravitational or centrifugal type, and the term in equation (4) due to temperature change is neglected. The particular solutions, u_i^p, for both the centrifugal and gravitational cases have been given by Henry *et al.* (1987) for axisymmetric bodies. When the complementary solution fields are substituted in equation (8), the domain integral terms drop out. The point q is taken, in the limit, to the boundary and the discretization of the boundary using boundary elements leads to the system of equations given as

$$[F]\{u\} = [G]\{t\} + [F]\{u^p\} - [G]\{t^p\}. \quad (10)$$

Differentiation of equation (10) with respect to the design variable X_L leads to the sensitivity equations as

$$[F]\{u\}_L = [G]_{,L}\{t\} + [G]\{t\}_{,L} - [F]_{,L}\{u\} + \{f^p\} \quad (11)$$

where

$$\{f^p\} = [F]\{u^p\}_{,L} + [F]_{,L}\{u^p\} - [G]\{t^p\}_{,L} - [G]_{,L}\{t^p\}. \quad (12)$$

In equation (11), the vectors $\{u\}$ and $\{t\}$ are known through the solution of equation (10). The vector $\{f^p\}$ due to the particular solution field is obtained from equation (12). The vectors $\{u^p\}_{,L}$ and $\{t^p\}_{,L}$ are obtained through the differentiation of the particular integral solution and are given as:

for gravitational loading:

$$u^p_{r,L} = -\frac{\nu\rho g}{E}(r_{,L}z + rz_{,L}),$$

$$u^p_{z,L} = \frac{\rho g}{E}(zz_{,L} + \nu rr_{,L}),$$

$$t^p_{r,L} = 0, \quad \text{and}$$

$$t^p_{z,L} = \rho g(z_{,L}n_z + zn_{z,L}) \quad (13)$$

for centrifugal loading:

$$
\begin{aligned}
u^{p}_{r,L} &= c_1[3(2+v/2)r^2 r_{,L}+(1-2v)(2zz_{,L}r+z^2 r_{,L})]\\
u^{p}_{z,L} &= -c_1(2nr_{,L}z+r^2 z_{,L})\\
t^{p}_{r,L} &= c_4[2(c_5 rr_{,L}+c_6 zz_{,L})n_r+(c_5 r^2+c_6 z^2)n_{r,L}+c_9(r_{,L}zn_z+rz_{,L}n_z+rzn_{z,L})]\\
t^{p}_{z,L} &= c_4[c_9(r_{,L}zn_z+rz_{,L}n_z+rzn_{r,L})-2(c_8 rr_{,L}+c_9 zz_{,L})n_z-(c_8 r^2+c_9 z^2)n_{z,L}]
\end{aligned}
\tag{14}
$$

where

$$
\begin{aligned}
c_1 &= -\rho\omega^2(1+v)(1-2v)/(8E(1-v))\\
c_4 &= -\rho\omega^2/8\\
c_5 &= (3-1.5v-v^2)/(1-v)\\
c_6 &= (1-2v)/(1-v)\\
c_8 &= (1-5v-2v^2)/(1-v)\\
c_9 &= -2v(1-2v)/(1-v).
\end{aligned}
\tag{15}
$$

ρ and ω are the mass density and the angular speed, respectively; n_r and n_z are the components of the normal in the r and z directions, respectively; and g is the acceleration due to gravity. All quantities needed for the solution of equation (11) to obtain the design sensitivities are then known.

The matrices $[F]$ and $[G]$ are the axisymmetric fundamental solution matrices. Their corresponding sensitivities are denoted by $[F]_{,L}$ and $[G]_{,L}$, respectively. The non-singular terms of these matrices are obtained by the use of appropriate Gauss quadrature rules whereas the singular terms are obtained by the use of boundary conditions corresponding to a rigid body motion and to an inflation mode. These details were discussed earlier in a paper by Saigal *et al.* (1989b).

Thermomechanical design sensitivity analysis

The volume integral in equation (8) which corresponds to the temperature variation ϕ can be reduced to a boundary integral through the use of Green's second identity as given by Bakr (1986). The resulting integral expression can be written as

$$
u_i(p) = -\int_\Gamma T_{ij}(p,q)u_j(q)\,\mathrm{d}\Gamma+\int_\Gamma U_{ij}(p,q)t_j(q)\,\mathrm{d}\Gamma + \frac{\alpha(1+v)}{8\pi(1-v)}\int_\Gamma\left\{\frac{\phi}{R}\left(n_i-R_{,i}\frac{\partial R}{\partial n}\right)-R_{,i}\frac{\partial\phi}{\partial n}\right\}\mathrm{d}\Gamma \tag{16}
$$

or in matrix form as

$$
\begin{Bmatrix}u_r(r,z)\\u_z(r,z)\end{Bmatrix} = \int_c\left\{\begin{bmatrix}G_{rr}G_{rz}\\G_{zr}G_{zz}\end{bmatrix}\begin{Bmatrix}t_r\\t_z\end{Bmatrix}-\begin{bmatrix}F_{rr}F_{rz}\\F_{zr}F_{zz}\end{bmatrix}\begin{Bmatrix}u_r\\u_z\end{Bmatrix}+\begin{bmatrix}V_{rr}V_{rz}\\V_{zr}V_{zz}\end{bmatrix}\begin{Bmatrix}\phi\\\partial\phi/\partial n\end{Bmatrix}\right\}\mathrm{d}c(R,Z). \tag{17}
$$

r, z are the boundary point coordinates; and R, Z are the integration point coordinates. In symbolic form, equation (17), after rearrangement is given by

$$
[F]\{u\} = [G]\{t\}+[V]\{T\}. \tag{18}
$$

The elasticity kernel matrices $[F]$ and $[G]$, and the thermoelastic kernel matrix $[V]$, may be found in the textbook by Bakr (1986). $\{u\}$ and $\{t\}$ are the nodal displacement and traction vectors, respectively. $\{T\}$ is the vector of nodal temperatures and temperature gradients. The implicit-differentiation of equation (18) leads to an expression for determining the design sensitivities of displacements and tractions. This expression is the same as given in equation (11). However, the vector $\{f^p\}$ is now given as

$$\{f^p\} = [V]_{,L}\{T\}+[V]\{T\}_{,L} \tag{19}$$

where

$$[V]_{,L} = \begin{bmatrix} V_{rr,L} V_{rz,L} \\ V_{zr,L} V_{zz,L} \end{bmatrix}.$$

The expressions for V_{rr}, V_{rz}, V_{zr} and V_{zz} were given by Bakr (1986) and involve the elliptic integrals of the first and the second kind, respectively. The derivatives of these elliptic integrals with respect to the design variable X_L are given by

$$\begin{aligned} E_{i,L} &= E_{i,m} m_{,L}; \quad i = 1,2 \\ m &= \frac{4rR}{Q^2} \\ m_{,L} &= \frac{4(r_{,L}R+rR_{,L})}{Q^2} - \frac{8rR[(r+R)(r_{,L}+R_{,L})+\bar{z}\bar{z}_{,L}]}{Q^4}, \\ Q^2 &= (r+R)^2+\bar{z}^2, \quad \bar{z} = z-Z \\ E_{1,m} &= -\frac{E_1}{m} + \frac{E_2}{m(1-m^2)} \\ E_{2,m} &= \frac{1}{m}(-E_1+E_2). \end{aligned} \tag{20}$$

The vector $\{f^p\}$ can now be calculated using these quantities in the differentiated expressions for $V_{rr,L}$, $V_{rz,L}$, $V_{zr,L}$ and $V_{zz,L}$. These expressions are listed in Appendix A.

Solution procedure and stress recovery

Starting with the initial configuration of the axisymmetric object, the unknown displacements $\{u\}$ and tractions $\{t\}$ are obtained from equation (10) for gravitational and centrifugal type body forces or from equation (18) for thermal type body forces. The initial configuration is then perturbed through a change in the design variable. The sensitivities of the geometric quantities ($r_{,L}$, $z_{,L}$, $n_{r,L}$, $n_{z,L}$, etc.) are obtained by applying forward-difference relationships to the initial and the perturbed configurations. These quantities are required for the evaluation of various terms in equation (11). The geometric sensitivities can also be obtained through analytical differentiation if the geometry of the changing region of the object can be expressed as a function of the design variable. This is usually possible for simple geometries such as those solved in the present paper. The design sensitivites are obtained by solving equation (11) where the vector $\{f^p\}$ is calculated using equations (12) or (19) for centrifugal and gravitational body forces or thermal body forces, respectively. This solution yields the boundary traction sensitivities only. The stress sensitivities at other locations can be obtained from the following expressions

$$\begin{Bmatrix} \sigma_{rr} \\ \sigma_{zz} \\ \sigma_{rz} \end{Bmatrix}_{,L} = \begin{bmatrix} n_r^2 & n_z^2 & -2n_r n_z \\ n_z^2 & n_r^2 & 2n_r n_z \\ n_r n_z & -n_r n_z & (n_r^2-n_z^2) \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix}_{,L} \tag{21}$$

where

Table 1. Design sensitivity analysis of a cylindrical bar under self-weight

Location Z (inch)	Sensitivity			
	Displacement × 10^4		Axial stress × 10^{-3}	
	Exact	This study	Exact	This study
0.0	7.72800	7.7282	0.00000	0.0026
0.5	7.70868	7.7091	−0.11592	−0.1148
1.0	7.65072	7.6513	−0.23184	−0.2317
1.5	7.55412	7.5547	−0.34776	−0.3476
2.0	7.41888	7.4195	−0.46368	−0.4639
2.5	7.24500	7.2456	−0.57960	−0.5814
3.0	7.03248	7.0327	−0.69552	−0.6989
3.5	6.78132	6.7809	−0.81144	−0.8185
4.0	6.49152	6.4901	−0.92736	−0.9381
4.5	6.16308	6.1600	−1.10433	−1.0606
5.0	5.79600	5.7904	−1.15920	−1.1832
5.5	5.39028	5.3814	−1.27512	−1.3091
6.0	4.94592	4.9327	−1.39104	−1.4251
6.5	4.46292	4.4445	−1.50696	−1.5627
7.0	3.94128	3.9170	−1.62288	−1.6904
7.5	3.38100	3.3508	−1.73880	−1.8148
8.0	2.78208	2.7468	−1.85472	−1.9386
8.5	2.14452	2.1069	−1.97064	−2.0495
9.0	1.46832	1.4330	−2.08656	−2.1596
9.5	0.75348	0.7299	−2.20248	−2.2389
10.0	0.00000	*0.0000	−2.31840	−2.3244

* Prescribed boundary condition.

$$\sigma_{22,L} = \frac{\nu}{(1-\nu)}\sigma_{11,L} + \frac{E}{(1-\nu^2)}(e_{22,L} + \nu e_{\theta\theta,L})$$

$$e_{22,L} = -\frac{J_{,L}}{J^2}u_{2,\xi} + \frac{1}{J}u_{2,\xi L}$$

$$e_{\theta\theta,L} = \frac{1}{r^2}(u_{r,L}r - u_r r_{,L}). \tag{22}$$

The subscripts 1 and 2 correspond, respectively, to the normal (η) and tangential (ξ) directions of a coordinate system located at the surface. The major advantage of the analytical implicit-differentiation approach lies in the fact that the left-hand side coefficient matrix $[F]$ which needs to be factorized to determine the sensitivities $\{u\}_{,L}$ in equation (11) is the same as the left-hand side coefficient matrix for equation (10). Thus this matrix is factorized only once and saved for later re-use, resulting in substantial economy of computer time.

NUMERICAL RESULTS

A series of axisymmetric test examples were solved to demonstrate the design sensitivity formulations for body forces discussed in the above sections. All computations reported here were carried out on a RIDGE 3200 computer system at Worcester Polytechnic Institute. For the problems with gravitational or centrifugal body forces, the following data were used: modulus of elasticity, $E = 3 \times 10^7$ psi; Poisson's ratio, $\nu = 0.3$; mass density, $\rho = 6$ lbm in^{-3}; angular rotation, $\omega = 10$ rad s^{-1} acceleration due to gravity, $g = 386.4$ lb-in s^{-2}.

(1) *Cylindrical bar under self weight*

A solid cylindrical bar of radius $r = 4$ inch and length, $l = 10$ inch, hanging in its gravitational field was studied. An axisymmetric section of the bar was modeled using 18 equal boundary elements. The axis of rotation of the bar was not modeled since appropriate fundamental solutions given by Bakr (1986) for this case were used. The length, l, of the bar was chosen as the design variable for which the sensitivities are required. The sensitivities for displacements and axial stress along the outer radius of the bar are given in Table 1. The location $z = 0$ corresponds to the free end of the bar and the bar was suspended at the location $z = 10$ inch. An analytical solution for response for this problem was given by Love (1944). The exact sensitivities for this case were obtained from the differentiation of the analytical solution for comparison. A good agreement of results along the entire length of the bar can be seen.

(2) *Solid sphere under rotation*
The design sensitivities for a solid sphere of radius 10 inch subjected to an angular velocity of 10 rad s^{-1} were obtained next. An analytical solution for the response of the sphere under rotation was given by Love (1944). The exact sensitivity data were obtained by the differentiation of this solution. The outer radius of the sphere was taken as the design variable and a perturbation of 0.001 inch was used to determine the geometric sensitivity quantities. A quarter of the axisymmetric section was modeled using 10 and 20 boundary elements, respectively, to model each of the straight and the curved sides of this section. The axis of the sphere was not modeled and appropriate fundamental solutions were used. The displacement and axial stress sensitivities along a radial axis are given in Table 2. The exact sensitivity results for this case are also reported in this table. A good agreement of the present results with the exact solution can be seen for both meshes used.

(3) *Rotating disk with hyperbolic varying thickness*
A hollow circular disk with a hyperbolic ($z = 10/r$) variation of thickness between the inner and the outer radii was analyzed. The inner radius of the disk was taken as the design variable. This case was considered since the geometry of the object is more general; the geometric sensitivity data due to perturbation of the inner radius are more general; and this perturbation includes the variation of normal at the nodal points in addition to the variation of their coordinates. The disk was subjected to an angular rotation of 10 rad s^{-1}.

A radial cross-section of the disk was modeled using 15 and 30 boundary elements, respectively. The element distribution for the 15 element model is shown in Fig. 1 and these elements were doubled for the 30 element model. The design variable was perturbed by 0.025% and the entire model was remeshed after the change in geometry. The sensitivity results for displacements and their convergence with increase in number of elements are given in Table 3. This table also gives the circumferential stress sensitivity results for the 30 element model. Analytical results were obtained from the elasticity solution given in Saada (1987) and are also shown in Table 3. These results are, however, based on the assumption of plane stress. With due regard to this assumption, a good agreement of the present results with the exact results can be observed.

(4) *Hollow plane strain cylinder under pressure and temperature variation*
A hollow cylinder under conditions of plane strain was analyzed. The ends of the

Table 2. Design sensitivity analysis of a solid sphere under rotation

Radius (inch)	Sensitivity: Displacement × 10^4 Exact	Mesh A	Mesh B	Axial stress × 10^{-2} Exact	Mesh A	Mesh B
0.00	0.00000	*0.0000	*0.0000	3.02521	2.9281	2.9302
1.00	1.71137	1.7114	1.7114	2.96471	2.9596	2.9651
2.00	3.38106	3.3810	3.3811	2.78319	2.7838	2.7839
3.00	4.96736	4.9673	4.9674	2.48067	2.4817	2.4813
4.00	6.42861	6.4286	6.4287	2.05714	2.0582	2.0577
5.00	7.72311	7.7231	7.7232	1.51261	1.5137	1.5131
6.00	8.80917	8.8091	8.8093	0.84706	0.8482	0.8474
7.00	9.64511	9.6451	9.6452	0.06050	0.0616	0.0607
8.00	10.1892	10.189	10.189	−0.84706	−0.8460	−0.8470
9.00	10.3999	10.400	10.400	−1.87563	−1.8772	−1.8759
10.00	10.2353	10.236	10.236	−3.02521	−3.0424	−3.0332

* Prescribed boundary condition.
Mesh A: 20 element mesh.
Mesh B: 40 element mesh.

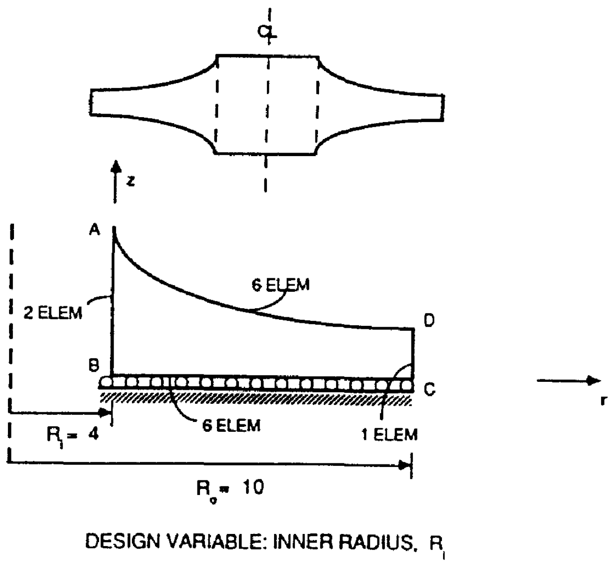

Fig. 1. Disk with varying thickness.

cylinder were thermally insulated and constrained not to move in the axial direction. The temperature field for the cylinder is then given as $T(r) = T_i - (T_i - T_o) \ln(r/R_i)/\ln(R_o/R_i)$ where T and R are the temperature and the radius, respectively; the subscripts i and o refer to the inner and the outer radius, respectively; r is the radius. The temperature and temperature gradients required for the evaluation of vector $\{f^p\}$ in equation (19) were calculated using this field. The numerical data used were as follows: $R_i = 3$, $R_o = 6$, $T_i = 5$, $T_o = 3$, $E = 1.0$, $\nu = 0.3$, and the coefficient of thermal expansion

Table 3. Design sensitivity analysis of a rotating disk with hyperbolic varying thickness

Radius (inch)	Sensitivity: Displacement × 10^1 Exact	Mesh A	Mesh B	Hoop str. × 10^{-3} Exact	Mesh B
4.00	2.05931	1.9987	1.9986	5.07485	3.9470
4.25	2.05819	—	1.9887	5.66207	5.4810
4.50	2.04760	1.9725	1.9725	5.99760	5.6433
4.75	2.02989	—	1.9522	6.16066	5.8450
5.00	2.00685	1.9296	1.9294	6.20577	5.8609
5.25	1.97984	—	1.9046	6.17095	5.8355
5.50	1.94995	1.8789	1.8785	6.08307	5.7574
5.75	1.91803	—	1.8512	5.96129	5.6542
6.00	1.88478	1.8236	1.8229	5.81938	5.5318
6.25	1.85078	—	1.7940	5.66731	5.4024
6.50	1.81653	1.7653	1.7645	5.51231	5.2668
6.75	1.78246	—	1.7348	5.35968	5.1372
7.00	1.74893	1.7060	1.7052	5.21325	5.0076
7.25	1.71628	—	1.6759	5.07583	4.8915
7.50	1.68481	1.6482	1.6474	4.94945	4.7787
7.75	1.65480	—	1.6198	4.83557	4.6831
8.00	1.62650	1.5945	1.5937	4.73521	4.5934
8.25	1.60016	—	1.5692	4.64908	4.5222
8.50	1.57602	1.5474	1.5466	4.57767	4.4589
8.75	1.55428	—	1.5261	4.52126	4.4141
9.00	1.53517	1.5089	1.5081	4.48003	4.3782
9.25	1.51889	—	1.4928	4.45403	4.3596
9.50	1.50565	1.4811	1.4803	4.44327	4.3511
9.75	1.49564	—	1.4710	4.44769	4.3540
10.00	1.48906	1.4660	1.4652	4.46718	4.4120

Note: exact solution is for the assumption of plane stress.
Mesh A: 15 element model; Mesh B: 30 element model.

Table 4. Design sensitivity analysis of a plane strain hollow cylinder under pressure and temperature variation

Radius (inch)	Sensitivity					
	Radial displacement		Radial stress		Circumferential stress	
	Exact	This study	Exact	This study	Exact	This study
3.0	3.6305	3.6315	0.000	0.000	0.5955	0.5956
3.5	3.6694	3.6712	−0.1058	−0.1074	0.6995	0.6993
4.0	3.6168	3.6174	−0.1181	−0.1186	0.7112	0.7110
4.5	3.5302	3.5309	−0.0971	−0.0975	0.6903	0.6902
5.0	3.4341	3.4350	−0.0653	−0.0654	0.6591	0.6594
5.5	3.3396	3.3402	−0.0318	−0.0318	0.6264	0.6265
6.0	3.2513	3.2523	0.000	0.000	0.5955	0.5956

$\alpha = 0.02$. The inner surface was also subjected to a pressure $P = 1.0$. The sensitivity of this design to changes in the inner radius R_i was studied.

A unit depth of the cylinder was modeled. A boundary element mesh of 16 elements, two along each of the axial sides and six along each of the radial sides, was used. The design sensitivities for radial displacement u_r, radial stress σ_r, and circumferential stress σ_θ are given in Table 4. The exact sensitivity results were computed by differentiating the response results given by Boley and Weiner (1960). The exact results are also shown in Table 4 and a good correlation of the present results with these exact results can be seen.

(5) *Pressurized hollow sphere under temperature variation*
A thick-walled hollow sphere of inner radius $R_i = 1.0$ and outer radius $R_o = 2.0$ was considered. The inner surface was maintained at temperature $T_i = 6.0$ and under pressure $P_i = 5.0$, while the outer surface was maintained at temperature $T_o = 2.0$ and pressure $P_o = 3.0$. The material properties used were: $E = 1.0$, $\nu = 0.3$, and $\alpha = 0.02$. The temperature field for the sphere is given by

$$T(r) = T_i + \frac{(T_i - T_o)R_o}{(R_o - R_i)}\left(\frac{R_i}{r} - 1\right)$$

where r is the radius. This temperature field was used in the expression for $\{f^p\}$ in equation (19). The inner radius R_i was taken as the design variable in this case. Due to double symmetry, only a quarter of the axisymmetric section was modeled. An exact solution for the response of the sphere was given by Boley and Weiner (1960) from which the sensitivity results were obtained by differentiation with respect to the inner radius R_i. Three different mesh distributions as shown in Fig. 2 were used to obtain both the response and the design sensitivity results. The results from the present analysis along with the analytical exact results are shown in Table 5 for the three meshes. A good comparison of results with a maximum error below 1% for displacement sensitivities with the refined mesh can be observed. Higher percentage errors for radial stress sensitivity were seen where the stress sensitivity had a value close to zero. It was noted from the results in Table 5 that as the mesh is refined, the design sensitivity analysis results show similar trends for convergence towards the exact results as the response results. This problem provides an example of the validity of the formulation for curved boundary elements and for treatment of the axis of rotational symmetry.

(6) *Solid sphere under constant temperature rise*
The sensitivity analysis of a solid sphere of radius $R_o = 2$ with respect to the outer radius R_o as design variable was considered next. The sphere was maintained at a uniform temperature of $T = 10$ and the numerical data used were: $E = 1.0$, $\nu = 0.3$ and $\alpha = 0.05$. Due to double symmetry of the axisymmetric section, only a quarter of this section was modeled using four boundary elements along the radial direction and two elements for the curved boundary. The analytical solution used for the previous example of hollow sphere can be used in this case with $R_i = 0$ to obtain the exact

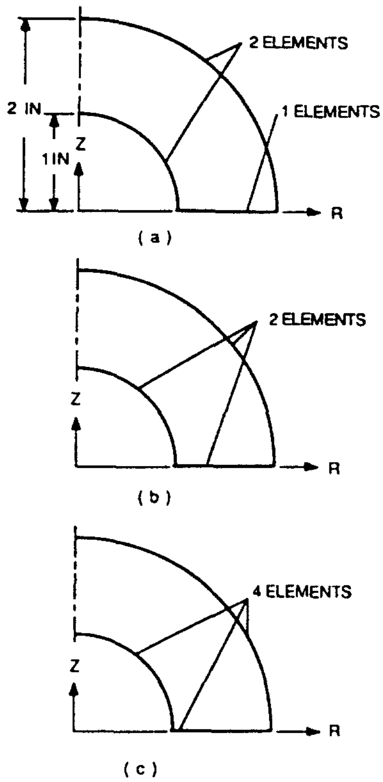

Fig. 2. Boundary element meshes for a pressurized hollow sphere under temperature variation.

Table 5. Design sensitivity analysis of a pressurized hollow sphere under temperature variation

Radius (inch)	Radial displacement				Radial displacement sensitivity			
	Exact	Five elements	Six elements	Twelve elements	Exact	Five elements	Six elements	Twelve elements
1.0	0.4628	0.4511	0.4569	0.4620	1.5126	1.4964	1.4991	1.5091
1.25	−0.3131	—	−0.3172	−0.3139	1.8563	—	1.8426	1.8535
1.50	−0.8586	−0.8559	−0.8616	−0.8593	1.9247	1.9090	1.9137	1.9222
1.75	−1.2954	—	−1.2978	−1.2959	1.9896	—	1.9807	1.9873
2.0	−1.6743	−1.6719	−1.6763	−1.6744	2.0996	2.0903	2.0946	2.0980

Radius (inch)	Radial stress				Circumferential stress			
	Exact	Five elements	Six elements	Twelve elements	Exact	Five elements	Six elements	Twelve elements
1.0	−5.00	−5.00	−5.00	−5.00	−1.653	−1.715	−1.660	−1.650
1.25	−3.902	—	−4.02	−3.86	−2.156	—	−2.190	−2.150
1.50	−3.407	−3.31	−3.35	−3.39	−2.373	−2.32	−2.360	−2.370
1.75	−3.149	—	−3.17	−3.14	−2.481	—	−2.550	−2.480
2.0	−3.00	−3.00	−3.00	−3.00	−2.539	−2.28	−2.530	−2.540

Radius (inch)	Radial stress sensitivity				Circumferential stress sensitivity			
	Exact	Five elements	Six elements	Twelve elements	Exact	Five elements	Six elements	Twelve elements
1.0	0.00	0.0	0.0	0.0	1.50	1.4730	1.498	1.4992
1.25	−0.901	—	−0.6667	−0.9384	1.923	—	1.9725	1.9151
1.50	−0.653	−0.4734	−0.6210	−0.6482	1.801	1.767	1.8044	1.8008
1.75	−0.297	—	−0.3007	−0.2908	1.634	—	1.6265	1.6341
2.0	−0.0	−0.0	0.0	0.0	1.50	1.4934	1.4967	1.5001

Table 6. Design sensitivity analysis of a solid sphere under uniform tmperature rise

Radius (inch)	Sensitivity					
	Radial displacement		Radial stress		Circumferential stress	
	Exact	This study	Exact	This study	Exact	This study
0.0	0.00	0.000	0.00	0.0029	0.00	−0.0027
0.25	0.0625	0.0625	0.00	0.0001	0.00	0.0001
0.50	0.125	0.1250	0.00	0.0001	0.00	−0.0001
0.75	0.1875	0.1876	0.00	0.0000	0.00	0.0000
1.00	0.250	0.2501	0.00	0.0000	0.00	0.0000
1.25	0.3125	0.3126	0.00	0.0000	0.00	0.0000
1.50	0.375	0.3752	0.00	0.0000	0.00	0.0000
1.75	0.4375	0.4378	0.00	−0.0002	0.00	0.0000
2.00	0.50	0.5007	0.00	0.0008	0.00	0.0012

sensitivity results. The sensitivity results from the present formulation are given in Table 6 for the displacement, radial stress, and circumferential stress sensitivities, respectively. The corresponding exact results are also shown in this table for comparison and a good agreement can be observed. The zero stress sensitivities along both the radial and the circumferential directions were accurately predicted up to the fourth digit as seen from the results in Table 6.

(7) *Compound hollow sphere under thermal stresses*

A compound sphere consisting of two thick-walled spheres perfectly bonded to each other was considered, as shown in Fig. 3. The sphere had an inner radius $R_1 = 1.0$, an interface radius $R_2 = 1.5$, and an outer radius $R_3 = 2.0$. The inner surface of the inner sphere of conductivity $k_1 = 1.0$ was under a pressure $P_1 = 5.0$ and maintained at a temperature $T_1 = 5.0$, while the outer surface of the outer sphere of conductivity $k_2 = 2.0$ was under a pressure $P_2 = 3.0$ and maintained at a temperature $T_2 = 3.0$. The material data were: $E_1 = 1.0$, $\nu_1 = 0.3$, $\alpha_1 = 0.15$ for the inner sphere and $E_2 = 2.0$, $\nu_2 = 0.2$, $\alpha_2 = 0.25$ for the outer sphere.

Due to symmetry about the $z = 0$ plane, only a quarter of the section of the compound sphere is modeled using 16 elements and two zones, as shown in Fig. 3. Zone I had 10 elements and was used to model the region of material 1 and Zone II had 10 elements and was used to model the region of material 2. Across the zone interface, the temperatures were continuous while the temperature gradients were discontinuous. A very accurate comparison of response results was obtained with the analytical solution given in the text by Bakr (1986).

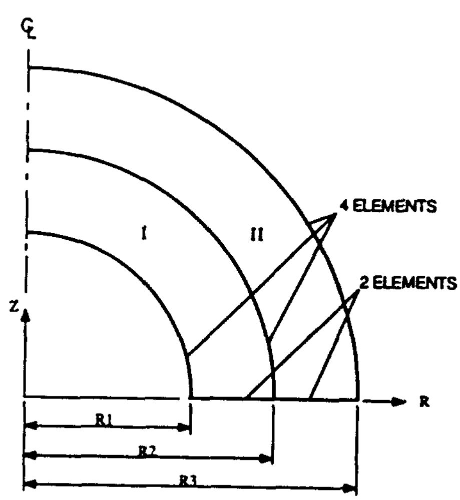

Fig. 3. Pressurized compound hollow sphere under thermal stresses.

Table 7. Design sensitivity analysis of a pressurized compound hollow sphere under temperature variation

Radius (inch)	Sensitivity: Displacement FDSA*	Displacement IDSA	Radial stress FDSA*	Radial stress IDSA	Circumferential stress FDSA*	Circumferential stress IDSA
1.0	1.370	1.4008	0.0	0.0	0.690	0.6597
1.125	1.622	1.6686	−0.820	−0.8146	1.010	1.0499
1.25	1.728	1.7551	−1.260	−0.9861	1.140	1.1295
1.375	1.739	1.8487	−1.30	−1.0836	1.150	1.3196
1.50 (inner)	1.762	1.7859	−1.23	−1.1182	1.210	1.1264
1.50 (outer)	1.762	1.7859	−1.23	−1.1182	2.570	2.497
1.625	1.682	1.7183	−0.81	−1.064	2.330	2.3912
1.75	1.662	1.6693	−0.48	−0.725	2.210	2.2655
1.875	1.650	1.6849	−0.23	−0.068	2.130	2.2503
2.0	1.631	1.6733	0.0	0.0	2.060	2.1664

FDSA = Finite Difference Sensitivity Analysis.
IDSA = Implicit Differentiation Sensitivity Analysis.
* Perturbation step size = 0.01 inch.

For the design sensitivity analysis, the inner radius R_i was considered as the design variable. The design sensitivities for radial displacements, radial stresses and circumferential stresses along the radius of the sphere are shown in Table 7. These sensitivities were also obtained using a finite-difference approach. In this approach two stress analysis were performed, first with the initial value of the design variable and next with a perturbed value of the design variable. A forward-difference relationship was then used for these two analyses to determine the required sensitivities. The sensitivity results for a perturbation of 0.01 of the inner radius of the present sphere are shown in Table 7. The two sets of results are in good agreement. It is noted that the finite-difference sensitivity results depend on the perturbation step size. This example was provided to demonstrate the applicability of the present developments to a class of practical problems for which the material parameters may vary spatially.

CONCLUDING REMARKS

The treatment of body forces of the centrifugal, gravitational, and thermal types in the implicit-differentiation formulation for the design sensitivity analysis of axisymmetric bodies using the boundary element method is presented. The particular integral sensitivity expressions for the gravitational and centrifugal type body forces are developed. The thermoelastic sensitivity kernels are given for the thermal type body forces. A wide range of problems dealing with a variety of axisymmetric bodies are presented. The sensitivities due to centrifugal, gravitational, and thermal body forces are solved and compared with the corresponding exact solutions. The accuracy of results obtained demonstrates the validity of the present formulation.

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APPENDIX A

The thermoelastic sensitivity kernels for equation (19) are given as

$$V_{rr_{,L}} = V_{rr1_{,L}}E_1 + V_{rr1}E_{1_{,L}} + V_{rr2_{,L}}E_2 + V_{rr2}E_{2_{,L}}$$
$$V_{zr_{,L}} = V_{zr1_{,L}}E_1 + V_{zr1}E_{1_{,L}} + V_{zr2_{,L}}E_2 + V_{zr2}E_{2_{,L}}$$
$$V_{rz_{,L}} = V_{rz1_{,L}}E_1 + V_{rz1}E_{1_{,L}} + V_{rz2_{,L}}E_2 + V_{rz2}E_{2_{,L}}$$
$$V_{zz_{,L}} = V_{zz1_{,L}}E_1 + V_{zz1}E_{1_{,L}}$$

where

$$V_{rr1_{,L}} = \frac{4c_1}{m^2c^3}\{2(rr_{,L} + RR_{,L})n_r + n_{r,L}(r^2+R^2) + (zR_{,L} + z_{,L}R)n_z + RzR_{z,L}\} - \frac{2mm_{,L}c^3 + 3m^2c^2c_{,L}}{m^2c^3}V_{rr1}$$

$$V_{rr2_{,L}} = \frac{4c_1}{m^2c^3D}\{(B_{,L}z^2 + 2Bzz_{,L} - D_{,L})n_r + n_{r,L}(Bz^2 - D) - [(R_{,L}z + Rz_{,L})H - RzH_{,L}]n_z - n_{z,L}RHz\}$$
$$-[2mm_{,L}c^3D + 3c^2c_{,L}m^2D + m^2c^3D_{,L}]\frac{V_{rr2}}{m^2c^3D}$$

$$V_{zr1_{,L}} = \frac{4c_1}{m^2c^3}(H_{,L}R + R_{,L}H) - \frac{1}{m^2c^3}(2mm_{,L}c^3 + 3c^2c_{,L}m^2)V_{zr1}$$

$$V_{zr2_{,L}} = -\frac{4c_1}{(m^2c^3)^2}(2mm_{,L}c^3 + 3c^2c_{,L}m^2) = -\frac{4c_1(2cm_{,L} + 3mc_{,L})}{m^3c^4}$$

$$V_{rz1_{,L}} = \frac{c_1}{RC}\{-z_{,L}n_r - n_{r,L}z + 2(R_{,L}n_z + Rn_{z,L})\} - \frac{1}{RC}(R_{,L}c + Rc_{,L})V_{rz1}$$

$$V_{rz2_{,L}} = \frac{c_1}{RCD}\{(R_{,L}z - Fz_{,L})n_r + n_{r,L}Fz - (4zz_{,L}R + 2z^2R_{,L})n_z - 2n_{z,L}z^2R\} - \frac{1}{RCD}(R_{,L}CD + C_{,L}RD + D_{,L}RC)V_{rz2}$$

$$V_{zz1_{,L}} = 2c_1\frac{(zc_{,L} - z_{,L}c)}{c^2}$$

where

$$B_{,L} = 2(rr_{,L} + RR_{,L} + zz_{,L})$$
$$C_{,L} = \frac{1}{c}[(r+R)(r_{,L} + R_{,L}) + zz_{,L}]$$
$$D_{,L} = 2[(R-r)(R_{,L} - r_{,L}) + zz_{,L}]$$
$$F_{,L} = 2(rr_{,L} - RR_{,L} + zz_{,L})$$
$$H_{,L} = 2(RR_{,L} - rr_{,L} + zz_{,L})$$
$$m_{,L} = \frac{4}{c^2}(R_{,L}r + r_{,L}R - 2Rrcc_{,L}).$$

All other expressions and constants are given in Bakr (1986).